Existence of Nash Equilibria in 2-Player Simultaneous Games and Priority Games Proven in Isabelle

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ABSTRACT

In previous work, we have studied a very general formalism of two-player games relevant for applications such as model checking. We assume *games* in which *strategies* by the players lead to *outcomes* taken from a *finite* set, and each player strives for an outcome that is optimal according to his/her preferences. We have shown using the proof assistants Isabelle and Coq that if the game has a certain structure, then a Nash equilibrium exists; more precisely, any game can be abstracted by disregarding the preferences and simply saying that some outcomes are mapped to "win for player 1", all the others to "win for player 2". The particular structure we consider are those games for which *every* such abstraction leads to a game which has a determined winner.

Here, we contribute several continuations of the work and their Isabelle formalisations.

KEYWORDS

Game theory, proof assistants, Nash equilibrium

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INTRODUCTION

Game theory is an interdisciplinary research topic (mathematics, economics ...) with many applications. When game theory is applied to critical multi-agent systems, one may use proof assistants to prove correctness [1, 5, 7, 8, 10, 11, 13, 14, 16] .

In previous work [15], we have formalised a game-theoretic result, essentially [9, Lemma 2.4], both in Coq and Isabelle. The result is as follows: starting from a two-player game with finitely many outcomes, one may derive a game by rewriting each of these outcomes with either of two basic outcomes, namely that Player 1 wins or that Player 2 wins. If all ways of deriving such a *win/lose game* lead to a game where one player has a winning strategy, the original game has a Nash equilibrium (NE).

Here, we present three extensions and applications of this work using Isabelle:

(1) Proving formally the existence of *secure* equilibria.
(2) The result of [15] is weaker than [9, Lemma 2.4] in that we consider preferences that are strict partial orders instead of just acyclic binary relations. However, the result easily also applies to (merely) acyclic preferences.
(3) Since the main theorem of [15] transforms determinacy into existence of NE, we apply it to the positional determinacy of parity games, which has been formalized in Isabelle [4].

Game forms are the central concept of this work. They can be instantiated into games by providing preferences for the players. Then, the NEs are defined for games. The win/lose games and their winning strategies are a special case.

Definition 1. A *game form* is a tuple $\langle S_1, S_2, O, v\rangle$ where

- S_1 and S_2 are the strategies of Players 1 and 2, resp.,
- O is a nonempty set (of possible outcomes),
- $v : S_1 \times S_2 \to O$ is the outcome function that values the strategy profiles.

A game form endowed with two binary relations $\prec_1, \prec_2$ over O for each player (modeling her preference) is called a *game*.

A *win/lose game* is a game where $O = \{\mathit{True}, \mathit{False}\}$ and the preferences are $\mathit{False} \prec_1 \mathit{True}$ and $\mathit{True} \prec_2 \mathit{False}$.

A win/lose game such that one player has a winning strategy is said to be *determined*.

For $i \in \{1, 2\}$, if Player i is the winning player and the winning strategy is in some $R_i \subseteq S_i$, the game is said to be *determined via* R_i.

Definition 2. Let $\langle S_1, S_2, O, v, \prec_1, \prec_2\rangle$ be a game. A strategy profile (s_1, s_2) in $S_1 \times S_2$ is a *Nash equilibrium* (NE) if it makes both players stable:

$$(\forall s_1' \in S_1,\ v(s_1, s_2) \not\prec_1 v(s_1', s_2))\wedge$$
$$(\forall s_2' \in S_2,\ v(s_1, s_2) \not\prec_2 v(s_1, s_2'))$$

Given a game form and a set $W \subseteq O$, one can *derive* a win/lose game in a straightforward way: player 1 wins iff the outcome is in W. If a game form is such that for *every* W, the derived win/lose game is determined, we call the game form itself *determined*.

In [15], we have formalised in Isabelle and Coq a theorem [9] stating that a game g whose game form is determined

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has an NE (note that this does not mean that g itself is determined, in general it is not even a win/lose game). Here is the **main theorem** in Isabelle code:

```
theorem equilibrium_transfer_finite :
  assumes finite0 : "finite (range (form g))"
      and trans1 : "trans (pref1 g)"
      and irref1 : "irrefl (pref1 g)"
      and trans2 : "trans (pref2 g)"
      and irref2 : "irrefl (pref2 g)"
      and det : "determinedForm (form g) R1 R2"
  shows "∃s1∈R1. ∃s2∈R2. isNash g s1 s2"
```

1 EXISTENCE AND COMPUTATION OF SECURE EQUILIBRIA

The secure equilibria [2, 3] of a game are the NEs of another game obtained by changing the usual preference of each player into a malevolent preference: instead of just trying to maximize her own payoff, she tries primarily to do so and, in case of ties, to minimize the opponent's payoff. In Isabelle code:

```
definition
  mal_pref :: "('a * 'a) set ⇒ ('a * 'a) set ⇒ ('a *
'a) set"
  where "mal_pref pr1 pr2 = pr1 ∪ (pr2⁻¹ - pr1⁻¹)"
```

Given a game, we can replace the preferences for each player by their respective malevolent extensions. We have a result corresponding to our main theorem above for this case. This extension is relatively straightforward and corresponds to approximately 250 lines of Isabelle proof script.

2 ACYCLIC PREFERENCES

Our main theorem assumes that the preferences are strict partial orders, i.e., transitive and irreflexive, and in this respect, it is weaker than [9, Lemma 2.4] where preferences are merely assumed to be acyclic. Recall that a relation r is acyclic if for all a, the pair (a, a) is not in r^+, the transitive closure of r.

Here we strengthen our main theorem by weakening the assumptions. Given a game with certain preferences, we can define the game obtained by replacing each player's preference by the respective transitive closure. By the main theorem, the transformed game has an NE, and we can show that this NE is also an NE of the original game by mere preference inclusion.

This extension is quite straightforward and corresponds to less than 100 lines of Isabelle proof script.

3 POSITIONAL DETERMINACY OF PARITY GAMES

The games above are *simultaneous* games. When we talk about "positional" and "parity games", we talk about *sequential* games: there is a graph partitioned so that each vertex is owned by one of the two players, and a play is a path through this graph. The path starts in some initial vertex, and in each vertex, the player owning it decides where to go next according to some strategy.

Definition 3. An *arena* is a tuple (V_1, V_2, v_0, E) where $V_1 \cap V_2 = \emptyset$, and $v_0 \in V := V_1 \cup V_2$, and $E \subseteq V^2$ is such that for all $v \in V$, the set $vE := \{u \in V \mid (v, u) \in E\}$ is non-empty.

A *positional strategy* of Player 1 in an arena (V_1, V_2, v_0, E) is a function $s : V_1 \to V$ such that $(v, s(v)) \in E$ for all $v \in V_1$.

The term "positional" refers to the fact that the strategy ignores the history of the path.

In a straightforward way, a strategy pair induces a unique infinite path (run, sequence of vertices) which we denote by $\langle s_1, s_2 \rangle$.

Definition 4 (Priority game form). A *priority game form* is an arena (V_1, V_2, v_0, E) together with a priority function $\pi : V \to \mathbb{N}$.

For an infinite path, the least priority occurring infinitely often as a label of a visited vertex is called *induced priority*.

Definition 5. A *win/lose priority game* consists of a priority game form and a subset $W \subseteq \mathbb{N}$. A run ρ is winning for Player 1 iff the induced priority of ρ is in W.

If $W := 2\mathbb{N}$, the win/lose priority game is called a *parity game*.

Definition 6. Given a priority game $(V_1, V_2, v_0, E, \pi, W)$, a Player 1 winning strategy is a Player 1 strategy s_1 such that for all Player 2 strategy s_2, the induced priority of $\langle s_1, s_2 \rangle$ is in W.

Dittmann [4] has shown in Isabelle that parity games are positionally determined (determinacy being defined similarly as for simultaneous games). The pen-and-paper proof was independently found by [5] and [12]. Based on a transformation from priority games to parity games, we can extend the statement to the priority games:

LEMMA 7. *Win/lose priority games with bounded π (i.e. involving finitely many priorities) are positionally determined.*

Note that Lemma 7, most likely a folklore result, could be proved by applying [6][Thm 2,Cor 7], but the proof that we formalize is more direct when assuming positional determinacy of parity games.

The Isabelle formalisation of this lemma with all the preliminaries comprises approximately 1300 lines of proof script, about as much as our entire Isabelle development of [15]. The difficulty is that infinite paths are defined coinductively, and thus statements relating different priority and parity games must be proven by coinduction.

So far we have considered parity games vs. the more general priority games. We can also link the sequential games of this section to the simultaneous games by putting a black box around the process of constructing an infinite sequence using strategies and then extracting a number from that infinite sequence. The black box is a game form in the simultaneous setting: it takes two strategies and returns a number.

At the same time, we can define *preference-priority games*, which are sequential games where rather than having a winning set W, we have preferences of the players on the outcomes in $\mathbb{N}$. By linking preference-priority games to the simultaneous setting, we can apply the main theorem above to show that preference-priority games also have an NE. We have done this on paper but the formal Isabelle development is work in progress.

In the future, we plan to present further extensions of [15], including also work concerning the proof assistant Coq.

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